On the definition of higher order differential operators in noncommutative geometry

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Abstract. Several definitions of differential operators on modules over a noncommutative ring are discussed.

Let $\mathcal{K}$ be a commutative ring and $\mathcal{A}$ a $\mathcal{K}$-ring (a unital algebra with $\mathbf{1} \neq 0$). Two-sided $\mathcal{A}$-modules throughout are assumed to be (central) bimodules over the center $\mathcal{Z}_\mathcal{A}$ of $\mathcal{A}$. Let P and Q be two-sided $\mathcal{A}$-modules. We discuss the notion of a $\mathcal{K}$-linear higher order Q-valued differential operator on P. If a ring $\mathcal{A}$ is commutative, there is the standard definition (up to an equivalence) of a differential operator on $\mathcal{A}$-modules [3, 5]. However, there exist its different generalizations to modules over a noncommutative ring [1, 2, 6]. At the same time, derivations of a noncommutative ring and the differential of a differential calculus over a noncommutative ring are defined in a standard way. It seems reasonable to regard them as differential operators.

Let $\mathcal{A}$ be a commutative $\mathcal{K}$-ring. Let P and Q be $\mathcal{A}$-modules (central bimodules). The $\mathcal{K}$-module $\mathrm{Hom}_\mathcal{K}(P,Q)$ of $\mathcal{K}$-linear homomorphisms $\Phi : P \to Q$ is endowed with the two different $\mathcal{A}$-module structures

$$(a\Phi)(p) := a\Phi(p), \qquad (\Phi \bullet a)(p) := \Phi(ap), \qquad a \in \mathcal{A}, \quad p \in P. \tag{1}$$

We will refer to the second one as the $\mathcal{A}^\bullet$-module structure. Let us put

$$\delta_a\Phi := a\Phi - \Phi \bullet a, \qquad a \in \mathcal{A}. \tag{2}$$

Definition 1. An element $\Delta \in \mathrm{Hom}_\mathcal{K}(P,Q)$ is an r-order Q-valued differential operator on P if $\delta_{a_0} \circ \cdots \circ \delta_{a_r}\Delta = 0$ for any tuple of $r+1$ elements $a_0, \ldots, a_r$ of $\mathcal{A}$.

This definition is equivalent to the following one.

Definition 2. An element $\Delta \in \mathrm{Hom}_\mathcal{K}(P,Q)$ is a zero order differential operator if $\delta_a\Delta = 0$ for all $a \in \mathcal{A}$, and Δ is a differential operator of order $r > 0$ if $\delta_a\Delta$ for all $a \in \mathcal{A}$ is an $(r-1)$-order differential operator.

In particular, zero order differential operators coincide with $\mathcal{A}$-module morphisms $P \to Q$. A first order differential operator Δ satisfies the condition

$$\delta_b \circ \delta_a \Delta(p) = ba\Delta(p) - b\Delta(ap) - a\Delta(bp) + \Delta(abp) = 0, \qquad \forall a, b \in \mathcal{A}. \tag{3}$$

The set $\mathrm{Diff}_r(P, Q)$ of r-order Q-valued differential operators on P inherits the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (1).

Remark 1. Let $P = \mathcal{A}$. Any zero order Q-valued differential operator Δ on $\mathcal{A}$ is uniquely defined by its value $\Delta(\mathbf{1})$. As a consequence, there is an isomorphism $\mathrm{Diff}_0(\mathcal{A}, Q) = Q$ via the association $Q \ni q \mapsto \Delta_q$, where Δ_q is given by the equality $\Delta_q(\mathbf{1}) = q$. A first order Q-valued differential operator Δ on $\mathcal{A}$ fulfils the condition

$$\Delta(ab) = b\Delta(a) + a\Delta(b) - ba\Delta(\mathbf{1}), \qquad \forall a, b \in \mathcal{A}.$$

It is a Q-valued derivation of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, i.e., the Leibniz rule

$$\Delta(ab) = \Delta(a)b + a\Delta(b), \qquad \forall a, b \in \mathcal{A}, \tag{4}$$

holds. The set $\mathfrak{d}(\mathcal{A}, Q)$ of derivations of $\mathcal{A}$ is an $\mathcal{A}$-module, but not an $\mathcal{A}^\bullet$-module. Any first order differential operator on $\mathcal{A}$ is split into the sum

$$\Delta(a) = a\Delta(\mathbf{1}) + [\Delta(a) - a\Delta(\mathbf{1})]$$

of the zero order differential operator $a\Delta(\mathbf{1})$ and the derivation $\Delta(a) - a\Delta(\mathbf{1})$. As a consequence, there is the $\mathcal{A}$-module decomposition

$$\mathrm{Diff}_1(\mathcal{A}, Q) = Q \oplus \mathfrak{d}(\mathcal{A}, Q).$$

If P and Q are two-sided $\mathcal{A}$-modules over a noncommutative $\mathcal{K}$-ring $\mathcal{A}$, the $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ can be provided with the left $\mathcal{A}$- and right $\mathcal{A}^\bullet$-module structures (1) and the similar right and left structures

$$(\Phi a)(p) := \Phi(p)a, \qquad (a \bullet \Phi)(p) := \Phi(pa), \qquad a \in \mathcal{A}, \qquad p \in P. \tag{5}$$

For the sake of convenience, we will refer to the $\mathcal{A} - \mathcal{A}^\bullet$ structures (1) and (5) as the left and right $\mathcal{A} - \mathcal{A}^\bullet$ structures, respectively. Let us put

$$\overline{\delta}_a \Phi := \Phi a - a \bullet \Phi, \qquad a \in \mathcal{A}, \qquad \Phi \in \mathrm{Hom}_{\mathcal{K}}(P, Q). \tag{6}$$

It is readily observed that $\delta_a \circ \overline{\delta}_b = \overline{\delta}_b \circ \delta_a$ for all $a, b \in \mathcal{A}$.

The left $\mathcal{A}$-module homomorphisms $\Delta : P \to Q$ obey the conditions $\delta_a \Delta = 0$, $\forall a \in \mathcal{A}$, and, consequently, they can be regarded as left zero order Q-valued differential operators on P. Similarly, right zero order differential operators are defined.

Utilizing the condition (3) as a definition of a first order differential operator in noncommutative geometry, one however meets difficulties. If $P = \mathcal{A}$ and $\Delta(\mathbf{1}) = 0$, the condition (3) does not lead to the Leibniz rule (4), i.e., derivations of the $\mathcal{K}$-ring $\mathcal{A}$ are not first order differential operators. In order to overcome these difficulties, one can replace the condition (3) with the following one [2].

Definition 3. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is called a first order differential operator on a two-sided module P over a noncommutative ring $\mathcal{A}$ if it obeys the relations

$$\begin{aligned} &\delta_a \circ \overline{\delta}_b \Delta = \overline{\delta}_b \circ \delta_a \Delta = 0, \qquad \forall a, b \in \mathcal{A}, \\ &a\Delta(p)b - a\Delta(pb) - \Delta(ap)b + \Delta(apb) = 0, \qquad p \in P. \end{aligned} \tag{7}$$

First order Q-valued differential operators on P make up a $\mathcal{Z}_{\mathcal{A}}$-module $\mathrm{Diff}_1(P, Q)$.

If P is a bimodule over a commutative ring $\mathcal{A}$, then $\delta_a = \overline{\delta}_a$ and Definition 3 comes to Definition 1 for first order differential operators.

Remark 2. Let $P = \mathcal{A}$. Any left or right zero order Q-valued differential operator Δ is uniquely defined by its value $\Delta(\mathbf{1})$. As a consequence, there are left and right $\mathcal{A}$-module isomorphisms

$$\begin{aligned} &Q \ni q \mapsto \Delta_q^{\mathrm{R}} \in \mathrm{Diff}_0^{\mathrm{R}}(\mathcal{A}, Q), \qquad \Delta_q^{\mathrm{R}}(a) = qa, \qquad a \in \mathcal{A}, \\ &Q \ni q \mapsto \Delta_q^{\mathrm{L}} \in \mathrm{Diff}_0^{\mathrm{L}}(\mathcal{A}, Q), \qquad \Delta_q^{\mathrm{L}}(a) = aq. \end{aligned}$$

A first order Q-valued differential operator Δ on $\mathcal{A}$ fulfils the condition

$$\Delta(ab) = \Delta(a)b + a\Delta(b) - a\Delta(\mathbf{1})b. \tag{8}$$

It is a derivation of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$. One obtains at once that any first order differential operator on $\mathcal{A}$ is split into the sums

$$\Delta(a) = a\Delta(\mathbf{1}) + [\Delta(a) - a\Delta(\mathbf{1})], \qquad \Delta(a) = \Delta(\mathbf{1})a + [\Delta(a) - \Delta(\mathbf{1})a]$$

of the derivations $\Delta(a) - a\Delta(\mathbf{1})$ or $\Delta(a) - \Delta(\mathbf{1})a$ and the left or right zero order differential operators $a\Delta(\mathbf{1})$ and $\Delta(\mathbf{1})a$, respectively. If u is a Q-valued derivation of $\mathcal{A}$, then au (1) and ua (5) are so for any $a \in \mathcal{Z}_{\mathcal{A}}$. Hence, Q-valued derivations of $\mathcal{A}$ constitute a $\mathcal{Z}_{\mathcal{A}}$-module $\mathfrak{d}(\mathcal{A}, Q)$. There are two $\mathcal{Z}_{\mathcal{A}}$-module decompositions

$$\mathrm{Diff}_1(\mathcal{A}, Q) = \mathrm{Diff}_0^{\mathrm{L}}(\mathcal{A}, Q) \oplus \mathfrak{d}(\mathcal{A}, Q), \qquad \mathrm{Diff}_1(\mathcal{A}, Q) = \mathrm{Diff}_0^{\mathrm{R}}(\mathcal{A}, Q) \oplus \mathfrak{d}(\mathcal{A}, Q).$$

They differ from each other in the inner derivations $a \mapsto aq - qa$.

Let $\mathrm{Hom}_{\mathcal{A}}^{\mathrm{R}}(P, Q)$ and $\mathrm{Hom}_{\mathcal{A}}^{\mathrm{L}}(P, Q)$ be the modules of right and left $\mathcal{A}$-module homomorphisms of P to Q, respectively. They are provided with the left and right $\mathcal{A} - \mathcal{A}^\bullet$-module structures (1) and (5), respectively.

Proposition 4. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is a first order Q-valued differential operator on P in accordance with Definition 3 iff it obeys the condition

$$\Delta(apb) = (\overrightarrow{\partial}\, a)(p)b + a\Delta(p)b + a(\overleftarrow{\partial}\, b)(p), \qquad \forall p \in P, \qquad \forall a, b \in \mathcal{A}, \tag{9}$$

where $\overrightarrow{\partial}$ and $\overleftarrow{\partial}$ are $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{A}}(P,Q)$- and $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{A}}(P,Q)$-valued derivations of $\mathcal{A}$, respectively. Namely, $(\overrightarrow{\partial}\, a)(pb) = (\overrightarrow{\partial}\, a)(p)b$ and $(\overleftarrow{\partial}\, b)(ap) = a(\overleftarrow{\partial}\, b)(p)$.

Proof. It is easily verified that, if Δ obeys the equalities (9), it also satisfies the equalities (7). Conversely, let Δ be a first order Q-valued differential operator on P in accordance with Definition 3. One can bring the condition (7) into the form

$$\Delta(apb) = [\Delta(ap) - a\Delta(p)]b + a\Delta(p)b + a[\Delta(pb) - \Delta(p)b],$$

and introduce the derivations

$$(\overrightarrow{\partial}\, a)(p) := \Delta(ap) - a\Delta(p), \qquad (\overleftarrow{\partial}\, b)(p) := \Delta(pb) - \Delta(p)b.$$

□

Remark 3. Let P be a differential calculus over a $\mathcal{K}$-ring $\mathcal{A}$ provided with an associative multiplication $\circ$ and a coboundary operator d. Then d exemplifies a P-valued first order differential operator on P by Definition 3. It obeys the condition (9) which reads

$$d(apb) = (da \circ p)b + a(dp)b + a((-1)^{|p|}p \circ db).$$

For instance, let $\mathfrak{d}\mathcal{A}$ be the Lie $\mathcal{K}$-algebra of $\mathcal{A}$-valued derivations of a $\mathcal{K}$-ring $\mathcal{A}$. Let us consider the Chevalley–Eilenberg complex of the Lie $\mathcal{K}$-algebra $\mathfrak{d}\mathcal{A}$ with coefficients in the ring $\mathcal{A}$, regarded as a $\mathfrak{d}\mathcal{A}$-module. This complex contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of $\mathcal{Z}_{\mathcal{A}}$-multilinear skew-symmetric maps $\phi : \overset{k}{\times}\, \mathfrak{d}\mathcal{A} \to \mathcal{A}$ with respect to the Chevalley–Eilenberg coboundary operator

$$d\phi(u_0, \ldots, u_k) = \sum_{i=0}^{k} (-1)^i u_i(\phi(u_0, \ldots, \widehat{u}_i, \ldots, u_k)) +$$
$$\sum_{i<j} (-1)^{i+j} \phi([u_i, u_j], u_0, \ldots, \widehat{u}_i, \ldots, \widehat{u}_j, \ldots, u_k).$$

Its terms $\mathcal{O}^k[\mathfrak{d}\mathcal{A}]$ are two-sided $\mathcal{A}$-modules. In particular,

$$(da)(u) = u(a), \qquad a \in \mathcal{A}, \qquad u \in \mathfrak{d}\mathcal{A}, \tag{10}$$
$$\mathcal{O}^1[\mathfrak{d}\mathcal{A}] = \mathrm{Hom}_{\mathcal{Z}_{\mathcal{A}}}(\mathfrak{d}\mathcal{A}, \mathcal{A}). \tag{11}$$

The graded module $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is provided with the product

$$\phi \wedge \phi'(u_1, ..., u_{r+s}) = \sum_{i_1 < \cdots < i_r; j_1 < \cdots < j_s} \mathrm{sgn}^{i_1 \cdots i_r j_1 \cdots j_s}_{1 \cdots r+s} \phi(u_{i_1}, \ldots, u_{i_r}) \phi'(u_{j_1}, \ldots, u_{j_s}).$$

This product obeys the relation

$$d(\phi\wedge\phi') = d(\phi)\wedge\phi' + (-1)^{|\phi|}\phi\wedge d(\phi'), \qquad \phi,\phi'\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}],$$

and makes $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ to a graded differential calculus over $\mathcal{A}$. This contains the graded differential subalgebra $\mathcal{O}^*\mathcal{A}$ generated by the elements da, $a\in\mathcal{A}$. One can show that

$$\mathfrak{d}\mathcal{A} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{O}^*\mathcal{A},\mathcal{A}). \tag{12}$$

In view of the relations (11) and (12), one can think of elements of $\mathcal{O}^1\mathcal{A}$ and $\mathfrak{d}\mathcal{A}$ in Remark 3 as being differential one-forms and vector fields in noncommutative geometry. A problem is that $\mathfrak{d}\mathcal{A}$ is not an $\mathcal{A}$-module. One can overcome this difficulty as follows [1].

Given a noncommutative $\mathcal{K}$-ring $\mathcal{A}$ and a two-sided $\mathcal{A}$-module Q, let d be a Q-valued derivation of $\mathcal{A}$. One can think of Q as being a first degree term of a differential calculus over $\mathcal{A}$. Let Q^*_{R} be the right $\mathcal{A}$-dual of Q. It is a two-sided $\mathcal{A}$-module:

$$(bu)(q) := bu(q), \qquad (ub)(q) := u(bq), \qquad \forall b\in\mathcal{A}, \qquad q\in Q.$$

One can associate to each element $u\in Q^*_{\mathrm{R}}$ the $\mathcal{K}$-module morphism

$$\widehat{u}:\mathcal{A}\ni a\mapsto u(da)\in\mathcal{A}. \tag{13}$$

This morphism obeys the relations

$$\widehat{(bu)}(a) = bu(da), \qquad \widehat{u}(ba) = \widehat{u}(b)a + \widehat{(ub)}(a). \tag{14}$$

One calls $(Q^*_{\mathrm{R}}, u\mapsto\widehat{u})$ the $\mathcal{A}$-right Cartan pair, and regards $\widehat{u}$ (13) as an $\mathcal{A}$-valued first order differential operator on $\mathcal{A}$ [1]. Note that $\widehat{u}$ (13) need not be a derivation of $\mathcal{A}$ and fails to satisfy Definition 3, unless u belongs to the two-sided $\mathcal{A}$-dual $Q^*\subset Q^*_{\mathrm{R}}$ of Q.

Morphisms $\widehat{u}$ (13) are called into play in order to describe (left) vector fields in noncommutative geometry [1, 4].

For instance, if $Q=\mathcal{O}^1\mathcal{A}$ in Remark 3, then au for any $u\in\mathfrak{d}\mathcal{A}$ and $a\in\mathcal{A}$ is a left noncommutative vector field in accordance with the relation (10).

Similarly the $\mathcal{A}$-left Cartan pair is defined. For instance, ua for any $u\in\mathfrak{d}\mathcal{A}$ and $a\in\mathcal{A}$ is a right noncommutative vector field.

If $\mathcal{A}$-valued derivation $u_1,\ldots u_r$ of a noncommutative $\mathcal{K}$-ring $\mathcal{A}$ or the above mentioned noncommutative vector fields $\widehat{u}_1,\ldots\widehat{u}_r$ on $\mathcal{A}$ are regarded as first order differential operators on $\mathcal{A}$, it seems natural to think of their compositions $u_1\circ\cdots u_r$ or $\widehat{u}_1\circ\cdots\widehat{u}_r$ as being particular higher order differential operators on $\mathcal{A}$. Turn to the general notion of a differential operator on two-sided $\mathcal{A}$-modules.

Let P and Q be regarded as left $\mathcal{A}$-modules [6]. Let us consider the $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ provided with the left $\mathcal{A}-\mathcal{A}^\bullet$ module structure (1). We denote $\mathcal{Z}_0$ its center, i.e., $\delta_a\Phi = 0$ for all $\Phi \in \mathcal{Z}_0$ and $a \in \mathcal{A}$. Let $\mathcal{I}_0 = \overline{\mathcal{Z}}_0$ be the $\mathcal{A}-\mathcal{A}^\bullet$ submodule of $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ generated by $\mathcal{Z}_0$. Let us consider: (i) the quotient $\mathrm{Hom}_{\mathcal{K}}(P,Q)/\mathcal{I}_0$, (ii) its center $\mathcal{Z}_1$, (iii) the $\mathcal{A}-\mathcal{A}^\bullet$ submodule $\overline{\mathcal{Z}}_1$ of $\mathrm{Hom}_{\mathcal{K}}(P,Q)/\mathcal{I}_0$ generated by $\mathcal{Z}_1$, and (iv) the $\mathcal{A}-\mathcal{A}^\bullet$ submodule $\mathcal{I}_1$ of $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ given by the relation $\mathcal{I}_1/\mathcal{I}_0 = \overline{\mathcal{Z}}_1$. Then we define the $\mathcal{A}-\mathcal{A}^\bullet$ submodules $\mathcal{I}_r$, $r = 2,\ldots$, of $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ by induction as follows:

$$\mathcal{I}_r/\mathcal{I}_{r-1} = \overline{\mathcal{Z}}_r, \tag{15}$$

where $\overline{\mathcal{Z}}_r$ is the $\mathcal{A}-\mathcal{A}^\bullet$ module generated by the center $\mathcal{Z}_r$ of the quotient $\mathrm{Hom}_{\mathcal{K}}(P,Q)/\mathcal{I}_{r-1}$.

Definition 5. Elements of the submodule $\mathcal{I}_r$ of $\mathrm{Hom}_{\mathcal{K}}(P,Q)$ are said to be left r-order Q-valued differential operators on a two-sided $\mathcal{A}$-module P [6].

Proposition 6. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is a differential operator of order r in accordance with Definition 5 iff it is brought into a finite sum

$$\Delta(p) = b_i\Phi^i(p) + \Delta_{r-1}(p), \qquad b_i \in \mathcal{A}, \tag{16}$$

where Δ_{r-1} and $\delta_a\Phi^i$ for all $a \in \mathcal{A}$ are left $(r-1)$-order differential operators if $r > 0$ and they vanish if $r = 0$.

Proof. If $r = 0$, the statement is a straightforward consequence of Definition 5. Let $r > 0$. The representatives Φ_r of elements of $\mathcal{Z}_r$ obey the relation

$$\delta_c\Phi_r = \Delta'_{r-1}, \qquad \forall c \in \mathcal{A}, \tag{17}$$

where Δ'_{r-1} is an $(r-1)$-order differential operator. Then representatives $\overline{\Phi}_r$ of elements of $\overline{\mathcal{Z}}_r$ take the form

$$\overline{\Phi}_r(p) = \sum_i c'_i\Phi^i(c_ip) + \Delta''_{r-1}(p), \qquad c_i, c'_i \in \mathcal{A},$$

where Φ^i satisfy the relation (17) and Δ''_{r-1} is an $(r-1)$-order differential operator. Due to the relation (17), we obtain

$$\overline{\Phi}_r(p) = b_i\Phi^i(p) + \Delta'''_{r-1}(p), \qquad b_i = c_ic'_i, \qquad \Delta'''_{r-1} = -\sum_i c'_i\delta_{c_i}\Phi^i + \Delta''_{r-1}. \tag{18}$$

Hence, elements of $\mathcal{I}_r$ modulo elements of $\mathcal{I}_{r-1}$ take the form (18), i.e., they are given by the expression (16). The converse is obvious. □

If $\mathcal{A}$ is a commutative ring, Definition 5 comes to Definition 2. Indeed, the expression (16) shows that $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P,Q)$ is an r-order differential operator iff $\delta_a\Delta$ for all $a \in \mathcal{A}$ is a differential operator of order $r-1$.

Proposition 7. The set $\mathcal{I}_r$ of r-order Q-valued differential operators on P is provided with the left and right $\mathcal{A}-\mathcal{A}^\bullet$ module structures.

Proof. This statement is obviously true for zero order differential operators. Using the expression (16), one can prove it for higher order differential operators by induction. □

Let $P=Q=\mathcal{A}$. Any zero order differential operator on $\mathcal{A}$ in accordance with Definition 5 takes the form

$$a\mapsto \sum_i c_i a c'_i, \qquad c_i, c'_i\in\mathcal{A}.$$

Proposition 8. Let Δ_1 and Δ_2 be n- and m-order $\mathcal{A}$-valued differential operators on $\mathcal{A}$, respectively. Then their composition $\Delta_1\circ\Delta_2$ is an $(n+m)$-order differential operator.

Proof. The statement is proved by induction as follows. If $n=0$ or $m=0$, it issues from the fact that the set of differential operators possesses both left and right $\mathcal{A}-\mathcal{A}^\bullet$ structures. Let us assume that $\Delta\circ\Delta'$ is a differential operator for any k-order differential operators Δ and s-order differential operators Δ' such that $k+s<n+m$. Let us show that $\Delta_1\circ\Delta_2$ is a differential operator of order $n+m$. Due to the expression (16), it suffices to prove this fact when $\delta_a\Delta_1$ and $\delta_a\Delta_2$ for any $a\in\mathcal{A}$ are differential operators of order $n-1$ and $m-1$, respectively. We have the equality

$$\begin{aligned}\delta_a(\Delta_1\circ\Delta_2)(b) &= a(\Delta_1\circ\Delta_2)(b)-(\Delta_1\circ\Delta_2)(ab) = \\ &\Delta_1(a\Delta_2(b))+(\delta_a\Delta_1\circ\Delta_2)(b)-(\Delta_1\circ\Delta_2)(ab)=(\Delta_1\circ\delta_a\Delta_2)(b)+(\delta_a\Delta_1\circ\Delta_2)(b),\end{aligned}$$

whose right-hand side, by assumption, is a differential operator of order $n+m-1$. □

Any derivation $u\in\mathfrak{d}\mathcal{A}$ of a $\mathcal{K}$-ring $\mathcal{A}$ is a first order differential operator in accordance with Definition 5. Indeed, it is readily observed that

$$(\delta_a u)(b)=au(b)-u(ab)=-u(a)b, \qquad b\in\mathcal{A},$$

is a zero order differential operator for all $a\in\mathcal{A}$. The compositions au, $u\bullet a$ (1), ua, $a\bullet u$ (5) for any $u\in\mathfrak{d}\mathcal{A}$, $a\in\mathcal{A}$ and the compositions of derivations $u_1\circ\cdots\circ u_r$ are also differential operators on $\mathcal{A}$ in accordance with Definition 5.

At the same time, noncommutative vector fields do not satisfy Definition 5 in general. First order differential operators by Definition 3 also need not obey Definition 5, unless $P=Q=\mathcal{A}$.

By analogy with Definition 5 and Proposition 6, one can define right differential operators on a two-sided $\mathcal{A}$-module P as follows.

Definition 9. Let P and Q be seen as right $\mathcal{A}$-modules over a noncommutative $\mathcal{K}$-ring $\mathcal{A}$. An element $\Delta\in\mathrm{Hom}_{\mathcal{K}}(P,Q)$ is said to be a right zero order Q-valued differential operator

on P if it is a finite sum $\Delta = \Phi^i b_i$, $b_i \in \mathcal{A}$, where $\overline{\delta}_a \Phi^i = 0$ for all $a \in \mathcal{A}$. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is called a right differential operator of order $r > 0$ on P if it is a finite sum

$$\Delta(p) = \Phi^i(p) b_i + \Delta_{r-1}(p), \qquad b_i \in \mathcal{A}, \tag{19}$$

where Δ_{r-1} and $\overline{\delta}_a \Phi^i$ for all $a \in \mathcal{A}$ are right $(r-1)$-order differential operators.

Definition 5 and Definition 9 of left and right differential operators on two-sided $\mathcal{A}$-modules are not equivalent, but one can combine them as follows.

Definition 10. Let P and Q be two-sided modules over a noncommutative $\mathcal{K}$-ring $\mathcal{A}$. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is a two-sided zero order Q-valued differential operator on P if it is either a left or right zero order differential operator. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is said to be a two-sided differential operator of order $r > 0$ on P if it is brought both into the form $\Delta = b_i \Phi^i + \Delta_{r-1}$, $b_i \in \mathcal{A}$, and $\Delta = \overline{\Phi}^i \overline{b}_i + \overline{\Delta}_{r-1}$, $\overline{b}_i \in \mathcal{A}$, where Δ_{r-1}, $\overline{\Delta}_{r-1}$ and $\delta_a \Phi^i$, $\overline{\delta}_a \overline{\Phi}^i$ for all $a \in \mathcal{A}$ are two-sided $(r-1)$-order differential operators.

One can think of this definition as a generalization of Definition 3 to higher order differential operators.

It is readily observed that two-sided differential operators described by Definition 10 need not be left or right differential operators, and *vice versa*. At the same time, $\mathcal{A}$-valued derivations of a $\mathcal{K}$-ring $\mathcal{A}$ and their compositions obey Definition 10.

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